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THE STABILITY OF COUETTE-TAYLOR ELECTROHYDRODYNAMIC FLOW*

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In the context of linear stability theory, an investigation is presented of the effect of electrohydrodynamic (EHD) interaction on the value of the critical Taylor number for loss of stability in a flow of a unipolarly charged fluid between rotating concentric cylinders in the presence of a radial electric field. The stabilizing effect of EHD interaction is demonstrated.

Previous studies /1/ have considered EHD instability of the equilibrium of a weakly conducting fluid between cylindrical electrodes, subjected to an injection of charge by one of the electrodes. Analogous problems have been solved for spherical electrodes /2, 3/. This paper investigates the possible effect of EHD interaction on the stability of a flow with curvilinear streamlines.

1. We consider the flow of a viscous incompressible unipolarly charged liquid between rotating concentric dielectric cylinders in the presence of a radial electric field. To fix our ideas we shall assume that the space charge density is positive. Attention will be confined to the case in which the outer cylinder is at rest. Let r^*, θ, z^* be cylindrical coordinates, R_1^* and R_2^* the radii of the inner and outer cylinders, respectively, and Ω^* the angular velocity of the inner cylinder. The asterisk indicates that the quantities in question are dimensional.

If the θ- and z-components of the electric field vector vanish, it follows from the equations of electrohydrodynamics /4/ that the velocity field of the fundamental unperturbed mode is independent of the presence of space charge. It is described by /5/

$$u_\theta^* = V^*(r^*) = A^*r^* + B^*/r^*$$
$$A^* = -\Omega^* \frac{\eta^2}{1-\eta^2}, \quad B^* = \Omega^* R_1^{*2} \frac{1}{1-\eta^2}, \quad \eta = \frac{R_1^*}{R_2^*} \tag{1.1}$$

The electrical parameters of the fundamental mode (space charge density and electrical field strength) are determined by means of Poisson's equation and the current continuity equation /4/. In the axisymmetric case, when $\partial j_\theta^*/\partial\theta = 0$, the latter reduces to the equation $j_r^* = 0$ (j_θ^* and j_r^* are the components of the current density vector).

The boundary conditions for the electrical parameters are formulated as follows. The radial component of the electric field on the outer surface of the inner cylinder, E_0^*, is assumed given. In this axisymmetric case, owing to the absence of eddies, the other field component vanishes: $E_\theta^* \equiv 0$. A field with this configuration may be created by an axisymmetrically distributed charge in the region $r^* \leqslant R_1^*$, e.g., by a uniformly charged metal cylinder with dielectrically coated outer surface $r^* = R_1^*$. In that case the electric field, having only a radial component, may be determined at $r^* = R_1^*$ from the charge distribution in the region $r^* \leqslant R_1^*$ using Gauss's Theorem. The electric field in the space between the cylinders is determined by the value of E_0^* and the space charge in that region and is independent of the charge on the outer cylinder.

Besides the boundary condition for the field, we also need a condition pertaining to the space charge density, which is given in this formulation of the problem in integral form as an expression for the electric current I^* per unit length due to charge transfer by the moving liquid.

Thus, the space charge density and electric field in the flow region can be determined by solving the following problem:

$$(r^*E^*)' = r^*Q^*/\varepsilon^*, \quad D^*Q^{*\prime} - b^*E^*Q^* = 0 \tag{1.2}$$
$$E^*(R_1^*) = E_0^*, \quad \int_{R_1^*}^{R_2^*} V^*Q^*\,dr^* = I^*$$

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Here the prime denotes differentiation with respect to r^*, $E^*(r^*)$ is the electric field, $Q^*(r^*)$ the space charge density, ε^* the permittivity of the liquid, and b^* and D^* the ion mobility and diffusivity. The equation $\operatorname{rot} \mathbf{E}^* = 0$, generally included in the system of equations of electrohydrodynamics /4/, is automatically satisfied here.

Introducing non-dimensional quantities

$$d^* = R_2^* - R_1^*, \quad R_0^* = {}^1/_2(R_2^* + R_1^*), \quad r^* = R_0^* + d^*x \qquad (1.3)$$
$$E^* = ED^*/(b^*d^*), \quad Q^* = QD^*\varepsilon^*/(b^*d^{*2})$$

and using the expression for the velocity of the liquid (1.1), we derive from (1.2) the following problem determining the electrical parameters of the flow:

$$(rE)' = rQ, \quad Q' - EQ = 0 \qquad (1.4)$$
$$r(x) = x + {}^1/_2\frac{1+\eta}{1-\eta}, \quad E(-{}^1/_2) = E_0$$
$$\int_{-1/2}^{1/2}\left[\frac{1}{r(1-\eta)^2} - r\right]Q\,dx = I, \quad I = \frac{b^*I^*}{\varepsilon^*D^*\Omega^*}\,\frac{1-\eta^2}{\eta^2}$$

The prime denotes differentiation with respect to x.

From (1.4), setting $\varphi(x) = r(x)E(x)$, we obtain the following equation:

$$r\varphi'' - \varphi' - \varphi\varphi' = 0$$

Integrating once, we obtain

$$\varphi' = [(\varphi + 2)^2 + C_1]/(2r) \qquad (1.5)$$

This equation can be integrated once more, but the form of the solution will depend on the sign of the constant C_1. If $C_1 = a^2$, then integration of (1.5) using the boundary Condition (1.4) gives the solution

$$E = \frac{a}{r}\operatorname{tg} X - \frac{2}{r}, \quad Q = \frac{1}{2}\left(\frac{a}{r}\right)^2\cos^{-2}X$$
$$X = \frac{a}{2}\ln\left(\frac{1-\eta}{\eta}r\right) + \operatorname{arctg}\frac{\eta E_0 + 2(1-\eta)}{(1-\eta)a}$$

On the other hand, if $C_1 = -a^2$, we have

$$E = \frac{a - 2 + (a+2)C_2r^a}{r(1 - C_2r^a)}, \quad Q = \frac{2a^2C_2r^{a-2}}{(1 - C_2r^a)^2}$$
$$C_2 = \left(\frac{1-\eta}{\eta}\right)^a\frac{2 - a + \eta(E_0 - 2 + a)}{2 + a + \eta(E_0 - 2 - a)}$$

Finally, if $C_1 = 0$,

$$E = -\frac{2(X+1)}{r}, \quad Q = \frac{2}{(rX)^2}, \quad X = [\eta E_0 + 2(1-\eta)] \times$$
$$\left\{2(1-\eta)\left[\ln\left(\frac{1-\eta}{\eta}r\right) - 1\right] + \eta E_0\ln\left(\frac{1-\eta}{\eta}r\right)\right\}^{-1}$$

The value of C_1 (or a) is determined from the second condition of (1.4), which can be written as

$$\frac{2}{(1-\eta)^2}\int_{-1/2}^{1/2}\frac{E}{r^2}\,dx = I + E_0\frac{1+\eta}{\eta}$$

2. The derive a system of equations and boundary conditions governing the evolution of small perturbations to the flow, we denote the perturbations to the r-, θ-, z-components of the velocity by $u^{*\prime}$, $v^{*\prime}$, $w^{*\prime}$, respectively. In addition , we let $p^{*\prime}$, $q^{*\prime}$, $e_r^{*\prime}$, $e_\theta^{*\prime}$, $e_z^{*\prime}$ denote the perturbations to pressure, space charge density and the field components, respectively.

We shall confine our attention to perturbations of the Taylor vortex type, which are axisymmetric, periodic in z^* and monotone in time /6/:

$$u^{*'}(r^*, z^*, t^*) = u^*(r^*)\exp(\sigma^* t^*)\cos(\alpha^* z^*) \text{ for } u^{*'}, v^{*'}, p^{*'}, q^{*'}, e_r^{*'}, e_\theta^{*'}$$
$$w^{*'}(r^*, z^*, t^*) = w^*(r^*)\exp(\sigma^* t^*)\sin(\alpha^* z^*) \text{ for } w^{*'}, e_z^{*'}$$

where σ^* and α^* are real numbers. The case of complex σ^* (i.e. oscillatory perturbations) will not be considered.

Boundary conditions: the velocity components are assumed to satisfy adherence conditions. The perturbations to the electrical parameters must be such that there is no current on the surface of either cylinder. In addition, the tangential component of the field must be continuous at the surface of the inner cylinder and the jump of the normal component must be equal to the space charge density. Since it is assumed that the field created by the charge distributed in the region $r^* < R_1^*$ is not perturbed, the tangential component of the field must vanish at $r^* = R_1^*$. The space charge density may be related to the space charge density at the surface, and therefore the boundary condition for the normal component of the field at $r^* = R_1^*$ will have the general form $e_r^{*'} + \beta^* q^{*'} = 0$, where β^* is a coefficient characterizing the adsorptive properties of the surface. In the sequel we shall assume that the dielectric surface of the cylinder does not adsorb surface charge, so that $\beta^* = 0$.

Substituting the above expressions for the perturbations into the electrohydrodynamic equations and omitting non-linear terms, we obtain a boundary-value problem describing the evolution of small perturbations of the type in question. Unlike the problem for the fundamental (unperturbed) mode, the equations for the electrical parameters are not separated from the "hydrodynamic part" of the problem - the hydrodynamic and electrical perturbations are interrelated.

The boundary-value problem for small perturbations may be written

$$L_1^* u^* + \alpha^* w^* = 0, \quad \rho^* \nu^* (L^* L_1^* - \alpha^{*2} - \sigma^*/\nu^*) u^* =$$
$$L^* p^* - 2\rho^* V^* v^*/r^* - E^* q^* - Q^* e_r^*$$
$$\nu^* (L^* L_1^* - \alpha^{*2} - \sigma^*/\nu^*) v^* = 2A^* u^*$$
$$\alpha^* p^* = -\rho^* \nu^* (L_1^* L^* - \alpha^{*2} - \sigma^*/\nu^*) w^* - Q^* e_z^*$$
$$\alpha^* q^* = -\varepsilon^* (L_1^* L^* - \alpha^{*2}) e_z^*, \quad D^* (L_1^* L^* - \alpha^{*2} - \sigma^*/D^*) q^* =$$
$$b^* E^* L^* q^* + 2b^* Q^* q^*/\varepsilon^* + (be_r^* + u^*) L^* Q^*$$
$$\alpha^* e_r^* + L^* e_z^* = 0, \quad e_\theta^* = 0$$
$$r^* = R_1^*, \quad u^* = v^* = w^* = e_r^* = e_z^* = (D^* L^* - b^* E^*) q^* = 0$$
$$r^* = R_2^*, \quad u^* = v^* = w^* = (D^* L^* - b^* E^*) q^* - b^* Q^* e_r^* = 0$$
$$(L^* = d/dr^*, \quad L_1^* = d/dr^* + 1/r^*)$$

To transform to non-dimensional notation, we use (1.3) and the relations

$$u^* = u\frac{\nu^* \Omega^* R_1^*}{2A^* d^{*2}} = -u\frac{\nu^*}{2d^*}\frac{1+\eta}{\eta}, \quad v^* = v\Omega^* R_1^*$$
$$\alpha^* = \frac{\alpha}{d^*}, \quad \sigma^* = \frac{\sigma\nu^*}{d^{*2}}, \quad e_z^* = \frac{eD^*}{b^* d^*}, \quad V^* = g(x)\Omega^* r^*$$
$$g(x) = \frac{\eta^2}{1-\eta^2}\left[\frac{4\xi^2(x)}{(1+\eta)^2} - 1\right], \quad \xi(x) = \frac{1+\eta}{1+\eta+2(1-\eta)x}$$

Omitting the equations of pressure, space charge density, the z-component of the velocity and the radial component of the field, and using Eqs.(1.4), we finally obtain the following system of ordinary differential equations and boundary conditions of tenth order:

$$(LL_1 - \alpha^2 - \sigma)(LL_1 - \alpha^2) u = -2Tg(x) v + \tag{2.1}$$
$$2\alpha\eta(1+\eta)^{-1}\lambda^{-2} NE (L_1 L - \alpha^2 - Q) e$$
$$(LL_1 - \alpha^2 - \sigma) v = u$$
$$(L_1 L - \alpha^2 - \lambda\sigma - 2Q)(L_1 L - \alpha^2) e - E(LL_1 - \alpha^2 + Q) Le =$$
$$\tfrac{1}{2}\alpha\lambda QE(1+\eta)\eta^{-1} u$$
$$T = 4\left(\frac{\Omega^* d^{*2}}{\nu^*}\right)^2 \frac{\eta^2}{1-\eta^2}, \quad \lambda = \frac{\nu^*}{D^*}, \quad N = \frac{\varepsilon^*}{\rho^* b^{*2}}$$
$$x = -\tfrac{1}{2}, \quad u = Lu = v = e = Le = 0$$
$$(L + 1 - \eta - E) L^2 e = 0$$
$$x = \tfrac{1}{2}, \quad u = Lu = v = 0$$

$$[LL_1L - EL_1L - (Q + \alpha^2)\, L + \alpha^2 E]\, e = 0$$
$$(L = d/dx,\ L_1 = d/dx + 1/r\,(x))$$

The function $r(x)$ is given by (1.4) and $E(x)$ and $Q(x)$ solutions of Problem (1.4).

3. The boundary-value Problem (2.1) can be solved by reduction to a Cauchy problem /7/. The solution is assumed to be a combination of four linearly independent solution, each satisfying the boundary conditions on the inner cylinder (summation from $i = 1$ to $i = 4$):

$$u = A_i u_i,\quad v = A_i v_i,\quad e = A_i e_i \tag{3.1}$$

Of the ten functions $u_i, Lu_i, L^2u_i, L^3u_i, v_i, Lv_i, e_i, Le_i, L^2e_i, L^3e_i$ occurring here, only the following are assumed to be non-zero at $x = -1/2$:

$$Lv_1 = 1,\quad L^2u_2 = 1,\quad L^3u_3 = 1,\quad L^2e_4 = 1,\quad L^3e_4 = E\,(-1/2) + (\eta - 1)/\eta \tag{3.2}$$

Eqs.(2.1) constitute a system of ten first-order equations in the above functions, which can be integrated by the Runge-Kutta method from $x = -1/2$ to $x = 1/2$ for each independent solution, taking the corresponding initial Conditions (3.2) into account. At $x = 1/2$ one obtains solutions of the form (3.1). The boundary Conditions (2.1) at $x = 1/2$ yield a system of homogeneous linear algebraic equations in the four independent constants A_i occurring in (3.1). The determinant of this system is the characteristic function of the eigenvalue Problem (2.1). Fixing σ and α one uses Newton's method to determine the eigenvalue of the Taylor number T which makes the characteristic function vanish.

We will now study the effect of EHD interaction on the critical Taylor number T_c for loss of stability. To determine T_c one takes $\sigma = 0$ and uses the above method to find the T values for a few equidistant values of α; T_c and α_c are then determined by quadratic interpolation from three values.

When solving the eigenvalue problem, the functions $E(x)$ and $Q(x)$ can be determined with the help of the above analytical expressions. However, some caution is necessary with regard to the sign of the constant C_1 in (1.5). For that reason, Eq.(1.5) should first be solved (before the eigenvalue problem), using the Runge-Kutta method with the same integration step-length; this yields profiles of $E(x)$, $Q(x)$, which can then be used to solve Problem (2.1).

4. The effect on stability of the space charge density and electric field distributions is of particular interest. These depend primarily on the parameters E_0 and I, and therefore the parameters λ and N in Problem (2.1), which characterize the liquid, were assumed constant and equal to unity in the computations reported below. The computations were also carried out for one value $\eta = 0.95$.

Fig.1 presents several parameters plotted against the electric current I: the constant C_1 in Eq.(1.5), the field strength E_e on the outer cylinder, and the space charge density on the inner cylinder (Q_0) and outer cylinder (Q_e) for fixed $E_0 = 0$. One observes that at $E_0 \geqslant 0$ the space charge density increases monotonically from the inner to the outer cylinder. At $E_0 < 0$, $Q(x)$ is not monotone until E_0 has decreased to such an extent that E_e becomes negative.

The functions plotted in Fig.1 characterize the electrical parameters of the flow to some extent. Fig.2 presents results of a stability computation for EHD flow: the current Taylor number T_c and the corresponding wave number α_c. The dotted and dash-dotted lines represent $T_c = 3509.0$ and $\alpha_c = 3.1276$, respectively, corresponding to Couette-Taylor flow without an impressed field. The data of Fig.2 indicate that at fixed $E_0 = 0$ an increase in the current (or total charge) between the cylinders first improves the stability of the flow. After T_c reaches its maximum value, approximately at $I = 8$, a further increase in charge causes a quite sharp loss of stability (represented by decreasing T_c), accompanied by an equally sharp increase in the wave number α_c.

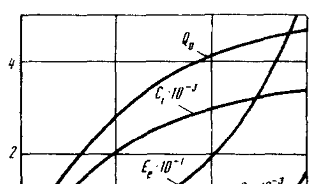

Fig.1

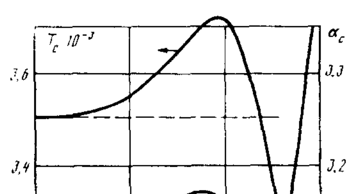

Fig.2

Fig.3 represents the effect of the impressed electric field at the inner cylinder on the values of the same parameters as in Fig.1, at a fixed value $I = 8$.

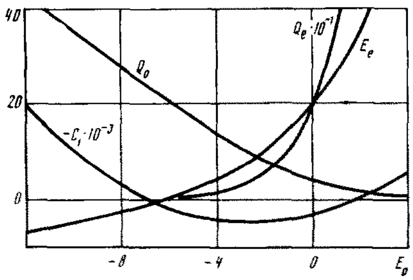

Fig.3

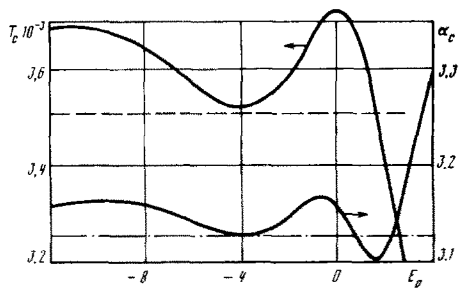

Fig.4

Fig.4 presents plots of T_c and α_c against E_0 for the same value of $I = 8$. The largest T_c value occurs at $E_0 \approx 0$. An increase in the negative values of the impressed field E_0 first causes a drop in the critical Taylor number, which reaches a local minimum at a distribution of electrical parameters which is almost symmetrical about the midpoint of the gap between the cylinders, as is evident on comparing Figs.3 and 4. At this point Q_0 and Q_e are equal, as are the absolute magnitudes of the charge gradient $dQ/dx = EQ$ about the cylinders. The minimum T_c and the corresponding wave number are almost the same as their values for ordinary Couette-Taylor flow.

When E_0 increases further, the flow becomes more stable and one observes a second flat maximum in the plots of T_c and α_c against E_0, with a monotone charge distribution, increasing toward the inner cylinder. At positive values of the impressed field E_0, an increase in which causes a sharp increase in charge density on the outer cylinder (Fig.3), there is a rapid decrease in stability with increasing wave number, similar to that observed when I is increased (Fig.2).

We may conclude that, although the maximum increase in stability of Couette-Taylor flow due to EHD interaction is observed when $E_0 = 0$, at which time the space charge increases monotonically from the inner to the outer cylinder, further concentration of charge on the concave surface of the outer cylinder, whether due to an increase in the impressed field or to the current in the gap between the cylinders (or the total charge), causes a rapid decrease in the critical Taylor number (destabilization of the flow) and an increase in the perturbation wave number. On the other hand, if the charage increases monotonically from the outer to the inner cylinder, one observes an increase in stability.

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